

RANK ONE NON-HERMITIAN PERTURBATIONS OF HERMITIAN β -ENSEMBLES OF RANDOM MATRICES

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ABSTRACT. For any $\beta > 0$, we provide a tridiagonal matrix model and compute the joint eigenvalue density of a random rank one non-Hermitian perturbation of Gaussian and Laguerre β -ensembles of random matrices.

1. INTRODUCTION

The energy Hamiltonian of a closed quantum system is usually modelled by a Hermitian random matrix H . The Hamiltonian of this system after coupling it to the outer world via s open channels is modelled in the physics literature by the so-called effective Hamiltonian¹

$$H_{eff} = H + i\Gamma, \quad (1.1)$$

where $\Gamma \geq \mathbf{0}$ is a rank s positive semi-definite Hermitian matrix that is independent of H . In this paper we are concerned with the exact joint eigenvalue distribution of (1.1) when there is one open channel ($\text{rank } \Gamma = s = 1$), and H is a Gaussian orthogonal/unitary/symplectic or Wishart orthogonal/unitary/symplectic random matrix. The law of Γ may be any continuous distribution, which is assumed to be given. We obtain tridiagonal models (in the spirit of Dumitriu–Edelman [DE02]) and compute the joint eigenvalue distribution for any $\beta > 0$, not merely $\beta = 1, 2, 4$.

Such ensembles are of active interest in the literature due to the numerous physical applications (see, e.g., the review papers [FS11, MRW10, FS03] and references therein).

The problem of computing the exact joint eigenvalue density of rank one non-Hermitian perturbations of Gaussian ensembles was considered in the physics literature in the papers of Ullah [Ull69] (for the case $\beta = 1$), Sokolov–Zelevinsky [SZ89] ($\beta = 1$), Stöckmann–Šeba [SŠ98] ($\beta = 1, 2$), Fyodorov–Khoruzhenko [FK99] ($\beta = 2$). The present paper provides a rigorous proof of this result (e.g., none of these papers addressed the question of the space of all attainable configurations of eigenvalues, which can be subtle, see below the case for Laguerre ensembles). Moreover, we obtain a generalization for any $\beta > 0$ and for any continuous distribution of Γ .

Let us also mention that the asymptotic analysis of these perturbations are also of high interest in the mathematics and physics literature and have been studied in [FS96, FS97, FS03, SFT99], see also [OW, Roc].

The joint eigenvalue density for rank one non-Hermitian perturbations of Wishart (Laguerre) ensembles has not appeared before neither in the mathematics nor

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¹In the physics literature it is more common to take $H - i\Gamma$, which can be reduced to our case by a simple symmetry.

physics literature. We treat all cases of $\beta > 0$, m , and n (we stress that cases $m < n$ and $m > n$ have drastically different behaviours here), and Γ .

The current paper is the Hermitian counterpart of the unitary results from [KK] (joint work with R. Killip). The important cornerstones in the proof are the Dumitriu–Edelman matrix models [DE02], and Arlinskiĭ–Tsekanovskii’s spectral analysis of (deterministic) Jacobi matrices with rank one imaginary part [AT06].

We note that our methods can provide matrix models (namely, *block* Jacobi matrices with independent (matrix-valued) Jacobi coefficients) for higher order perturbations $s \geq 2$ as well, which could prove to be useful for computing their eigenvalue density (for the case $\beta = 2$, $s \geq 2$, Fyodorov–Khoruzhenko [FK99] provide another approach). We leave this as a challenging open problem.

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2. PRELIMINARIES

2.1. Gaussian and Wishart ensembles.

Definition 1. We say that a real-valued random variable (r.v.) ξ is $N(0, \sigma)$ -distributed, and we write $\xi \sim N(0, \sigma)$, if its probability distribution function (p.d.f.) is $\frac{1}{\sqrt{2\pi}\sigma^2} e^{-x^2/2\sigma^2}$.

We say that a complex-valued r.v. ξ is $N(0, \sigma \mathbf{I}_2)$ -distributed (where \mathbf{I}_k is the $k \times k$ identity matrix) if $\operatorname{Re} \xi$ and $\operatorname{Im} \xi$ are independent and each distributed according to $N(0, \sigma)$.

We say that a quaternion-valued r.v. ξ is $N(0, \sigma \mathbf{I}_4)$ -distributed if $\xi = \xi_1 + \xi_2 i + \xi_3 j + \xi_4 k$ and ξ_1, \dots, ξ_4 are independent and each distributed according to $N(0, \sigma)$.

We say that a real-valued r.v. ξ is χ_k^2 distributed ($k > 0$) if its p.d.f. is $\frac{1}{2^{k/2} \Gamma(k/2)} x^{k/2-1} e^{-x/2}$. For integer $k > 0$ this can be realized by the sum of squares of k independent $N(0, 1)$ variables.

We say that a real-valued r.v. ξ is χ_k distributed ($k > 0$) if it can be realized as the square root of a χ_k^2 random variable. Its p.d.f. is $\frac{2^{1-k/2}}{\Gamma(k/2)} x^{k-1} e^{-x^2/2}$.

We say that a real-valued r.v. ξ is $\tilde{\chi}_k$ distributed ($k > 0$) if its p.d.f. is $\frac{2}{\Gamma(k/2)} x^{k-1} e^{-x^2}$ (this coincides with $\frac{1}{\sqrt{2}} \chi_k$ distribution).

Definition 2. Let Y be an $n \times n$ matrix with independent identically distributed (i.i.d.) entries chosen from $N(0, 1)$, $N(0, \mathbf{I}_2)$, or $N(0, \mathbf{I}_4)$. Then we say that $X = \frac{1}{2}(Y + Y^*)$ belongs to the Gaussian orthogonal/unitary/symplectic ensemble, respectively. We denote it by GOE_n , GUE_n , GSE_n , respectively.

Definition 3. Let Y be an $m \times n$ matrix with i.i.d. entries chosen from $N(0, 1)$, $N(0, \mathbf{I}_2)$, or $N(0, \mathbf{I}_4)$. Then we say that the $n \times n$ matrix $X = Y^* Y$ belongs to the Wishart orthogonal/unitary/symplectic ensemble, respectively. We denote it by $LOE_{(m,n)}$, $LUE_{(m,n)}$, $LSE_{(m,n)}$, respectively.

To avoid confusion, we stress that $LOE_{(m,n)}/LUE_{(m,n)}/LSE_{(m,n)}$ ensembles consist of $n \times n$ matrices.

2.2. Tridiagonalization of Hermitian matrices. Let H be an $n \times n$ Hermitian matrix. Let us describe a process that we will call the tridiagonalization procedure.

Denote \mathbf{e}_j to be the j -th standard vector in \mathbb{C}^n , that is, having 1 in its j -th entry and 0 everywhere else. Let $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^* \mathbf{y}$, the usual inner product in \mathbb{C}^n .

Let us apply the Gram–Schmidt orthogonalization procedure in \mathbb{C}^n to the sequence of vectors $\mathbf{e}_1, H\mathbf{e}_1, H^2\mathbf{e}_1, \dots, H^{k-1}\mathbf{e}_1$, where $k = \dim \text{span}\{H^j\mathbf{e}_1 : j \geq 0\}$. Note that $1 \leq k \leq n$. After normalization we obtain an orthonormal sequence of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in \mathbb{C}^n . If $k < n$, then we choose an arbitrary unit vector \mathbf{v}_{k+1} in $\mathbb{C}^n \ominus \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and repeat the procedure but with \mathbf{v}_{k+1} instead of \mathbf{e}_1 . By repeating this procedure finitely many times more if necessary and combining all the resulting vectors together, we obtain an orthonormal basis $\{\mathbf{v}_j\}_{j=1}^n$ of \mathbb{C}^n .

Standard arguments (see, e.g., [Sim11, Sect 1.3]) show that the matrix of H in the basis $\{\mathbf{v}_j\}_{j=1}^n$ is tridiagonal. In other words, if we form unitary matrix S with $\{\mathbf{v}_j\}_{j=1}^n$ as its columns, then $SHS^* = \mathcal{J}$, where

$$\mathcal{J} = SHS^* = \begin{pmatrix} b_1 & a_1 & 0 & & \\ a_1 & b_2 & a_2 & \ddots & \\ 0 & a_2 & b_3 & \ddots & 0 \\ & \ddots & \ddots & \ddots & a_{n-1} \\ & & 0 & a_{n-1} & b_n \end{pmatrix}, \quad a_j \geq 0, b_n \in \mathbb{R}. \quad (2.1)$$

We call matrices of the form (2.1) Jacobi, and the coefficients $\{a_j, b_j\}$ — their Jacobi coefficients. For a future reference, observe that

$$S\mathbf{e}_1 = S^*\mathbf{e}_1 = \mathbf{e}_1 \quad (2.2)$$

since $\mathbf{v}_1 = \mathbf{e}_1$ in the Gram–Schmidt procedure. Note that in the tridiagonalization procedure above, if $\dim \text{span}\{H^j\mathbf{e}_1 : j \geq 0\} = k < n$, then $a_k = 0$, i.e., \mathcal{J} becomes a direct sum of Jacobi matrices.

2.3. Matrix models for Gaussian and Wishart ensembles. Now let us apply the tridiagonalization procedure from the previous section to a random matrix from a Gaussian or a Wishart ensemble.

If H is from GOE_n , GUE_n , or GSE_n , then \mathbf{e}_1 is a cyclic vector for H with probability 1. Therefore we obtain (2.1) with $a_j > 0$ for all $1 \leq j \leq n-1$.

The same is true for a random matrix H from $LOE_{(m,n)}$, $LUE_{(m,n)}$, or $LSE_{(m,n)}$, but only if $m \geq n$. If $m < n$, then with probability 1, $\dim \text{span}\{H^j\mathbf{e}_1 : j \geq 0\} = m+1 \leq n$, and $\mathbb{C}^n \ominus \text{span}\{H^j\mathbf{e}_1 : j \geq 0\} \subseteq \ker H$, so that the resulting Jacobi matrix (2.1) that we obtain has $a_{m+1} = \dots = a_{n-1} = 0$, $b_{m+2} = \dots = b_n = 0$. In other words, we have that \mathcal{J} is the direct sum of an $(m+1) \times (m+1)$ Jacobi matrix and the $(n-m-1) \times (n-m-1)$ zero matrix. The proof of this case can be done by following the Dumitriu–Edelman [DE02] arguments.

Lemma 1 (Dumitriu–Edelman [DE02]). *Let H be a GOE_n , GUE_n , or GSE_n matrix. There exists a unitary matrix S satisfying (2.2) such that $SHS^* = \mathcal{J}$ is tridiagonal (2.1), where*

$$\begin{aligned} a_j &\sim \tilde{\chi}_{\beta(n-j)}, & 1 \leq j \leq n-1, \\ b_j &\sim N(0, 1), & 1 \leq j \leq n, \end{aligned}$$

where $\beta = 1, 2, 4$ for GOE_n , GUE_n , GSE_n , respectively.

Lemma 2 (Dumitriu–Edelman [DE02]). *Let H be a $LOE_{(m,n)}$, $LUE_{(m,n)}$, or $LSE_{(m,n)}$ matrix. There exists a unitary matrix S satisfying (2.2) such that $SHS^* = \mathcal{J} = B^*B$ is tridiagonal (2.1), where*

$$B = \begin{pmatrix} x_1 & y_1 & 0 & & \\ 0 & x_2 & y_2 & \ddots & \\ 0 & 0 & x_3 & \ddots & 0 \\ & \ddots & \ddots & \ddots & y_{n-1} \\ & & 0 & 0 & x_n \end{pmatrix}, \quad \text{with} \quad (2.3)$$

(i) If $m \geq n$:

$$\begin{aligned} x_j &\sim \chi_{\beta(m-j+1)}, & 1 \leq j \leq n, \\ y_j &\sim \chi_{\beta(n-j)}, & 1 \leq j \leq n-1; \end{aligned}$$

(ii) If $m \leq n-1$:

$$\begin{aligned} x_j &\sim \begin{cases} \chi_{\beta(m-j+1)}, & \text{if } 1 \leq j \leq m, \\ 0, & \text{if } m+1 \leq j \leq n, \end{cases} \\ y_j &\sim \begin{cases} \chi_{\beta(n-j)}, & \text{if } 1 \leq j \leq m, \\ 0, & \text{if } m+1 \leq j \leq n-1; \end{cases} \end{aligned}$$

where $\beta = 1, 2, 4$ for $LOE_{(m,n)}$, $LUE_{(m,n)}$, $LSE_{(m,n)}$, respectively.

Remarks. 1. For GSE_n and $LSE_{(m,n)}$ every entry is quaternionic, so all the instances of \mathbb{C} in the arguments above should be replaced with the algebra of quaternions. The resulting coefficients a_j , b_j , x_j , y_j in Lemmas 1, 2 are quaternionic too, but with the i , j , and k parts equal to zero.

2. We adopt a different notation from the one used in [DE02]: the roles of a_j 's and b_j 's are switched; the orderings of a_j , b_j , x_j , y_j have been reversed; Wishart ensembles are taken to be W^*W instead of WW^* .

2.4. Gaussian and Laguerre β -ensembles. The tridiagonal matrix ensembles from Lemmas 1 and 2 make sense for any $0 < \beta < \infty$, not merely for $\beta = 1, 2, 4$. We will call them Gaussian β -ensembles $G\beta E_n$ and Laguerre β -ensembles $L\beta E_{(m,n)}$, respectively.

2.5. Spectral measures of Gaussian and Laguerre β -ensembles. By the Riesz representation theorem, for any Hermitian matrix H there exists a probability measure μ satisfying

$$\langle \mathbf{e}_1, H^k \mathbf{e}_1 \rangle = \int_{\mathbb{R}} x^k d\mu(x), \quad \text{for all } k \geq 0. \quad (2.4)$$

We call μ the spectral measure of H corresponding to the vector \mathbf{e}_1 .

In fact, any Hermitian can be unitarily diagonalized, so that we can write $H = UDU^*$, where D is the diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ of H on the diagonal, and the columns $\mathbf{u}_1, \dots, \mathbf{u}_n$ of U are the corresponding orthonormal eigenvectors of H . This easily implies (2.4) with

$$\mu(x) = \sum_{j=1}^n w_j \delta_{\lambda_j}, \quad \text{where } w_j = |\langle \mathbf{e}_1, \mathbf{u}_j \rangle|^2. \quad (2.5)$$

Here δ_λ is the Dirac measure at λ , i.e., the probability measure concentrated at a point λ . Note that the support of μ consists of $\leq n$ points ($< n$ if some of the eigenvalues coincide or if some of the eigenvectors are orthogonal to \mathbf{e}_1).

Note that because of (2.2), the spectral measures of H and of its Jacobi form \mathcal{J} coincide, that is H and \mathcal{J} have identical eigenvalues λ_j 's and eigenweights w_j 's. In particular, spectral measures of GOE_n and $G\beta E_n$ with $\beta = 1$ coincide; spectral measures of GUE_n and $G\beta E_n$ with $\beta = 2$ coincide; *quaternion-valued* spectral measures of GSE_n and $G\beta E_n$ with $\beta = 4$ (viewed as a matrix with purely-real quaternion entries) coincide. Analogous statements can be made for $LOE_{(m,n)}/LUE_{(m,n)}/LSE_{(m,n)}$ and $L\beta E_{(m,n)}$.

We remark that all the statements in lemmas and theorems below should be understood to hold with probability 1.

Lemma 3 (Dumitriu–Edelman [DE02]). *For any $0 < \beta < \infty$, the spectral measure of $G\beta E_n$ -ensemble is (2.5) where $\lambda_1, \dots, \lambda_n, w_1, \dots, w_{n-1}$ are distributed on*

$$\sum_{j=1}^n w_j = 1; \quad w_j > 0, \quad 1 \leq j \leq n; \quad \lambda_j \in \mathbb{R} \quad (2.6)$$

according to

$$\frac{1}{g_{\beta,n}} \prod_{j=1}^n e^{-\lambda_j^2/2} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^\beta d\lambda_1 \dots d\lambda_n \times \frac{1}{c_{\beta,n}} \prod_{j=1}^n w_j^{\beta/2-1} dw_1 \dots dw_{n-1}, \quad (2.7)$$

where

$$g_{\beta,n} = (2\pi)^{n/2} \prod_{j=1}^n \frac{\Gamma(1 + \beta j/2)}{\Gamma(1 + \beta/2)}, \quad c_{\beta,n} = \frac{\Gamma(\beta/2)^n}{\Gamma(\beta n/2)}. \quad (2.8)$$

Lemma 4 (Dumitriu–Edelman [DE02]). *For any $m \geq n$ and any $0 < \beta < \infty$, the spectral measure of $L\beta E_{(m,n)}$ -ensemble is (2.5) where $\lambda_1, \dots, \lambda_n, w_1, \dots, w_{n-1}$ are distributed on*

$$\sum_{j=1}^n w_j = 1; \quad w_j > 0, \quad 1 \leq j \leq n; \quad \lambda_j > 0 \quad (2.9)$$

according to

$$\frac{1}{l_{\beta,n,a}} \prod_{j=1}^n \lambda_j^{\beta a/2} e^{-\lambda_j/2} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^\beta d\lambda_1 \dots d\lambda_n \times \frac{1}{c_{\beta,n}} \prod_{j=1}^n w_j^{\beta/2-1} dw_1 \dots dw_{n-1}, \quad (2.10)$$

where $a = |m - n| + 1 - 2/\beta$, and

$$l_{\beta,n,a} = 2^{n(a\beta/2+1+(n-1)\beta/2)} \prod_{j=1}^n \frac{\Gamma(1 + \beta j/2) \Gamma(1 + \beta a/2 + \beta(j-1)/2)}{\Gamma(1 + \beta/2)}, \quad (2.11)$$

and $c_{\beta,n}$ is as in (2.8).

Proposition 1. *For any $m \leq n - 1$ and any $0 < \beta < \infty$, the spectral measure of $L\beta E_{(m,n)}$ -ensemble is*

$$\mu(x) = w_0 \delta_0 + \sum_{j=1}^m w_j \delta_{\lambda_j}, \quad (2.12)$$

where $\lambda_1, \dots, \lambda_m, w_1, \dots, w_m$ are distributed on

$$\sum_{j=0}^m w_j = 1; \quad w_j > 0, \quad 0 \leq j \leq m; \quad \lambda_j > 0 \quad (2.13)$$

according to

$$\begin{aligned} & \frac{1}{l_{\beta, m, a}} \prod_{j=1}^m \lambda_j^{\beta a/2} e^{-\lambda_j/2} \prod_{1 \leq j < k \leq m} |\lambda_j - \lambda_k|^\beta d\lambda_1 \dots d\lambda_m \\ & \times \frac{1}{d_{\beta, m, n}} w_0^{\beta(n-m)/2-1} \prod_{j=1}^m w_j^{\beta/2-1} dw_1 \dots dw_m, \end{aligned} \quad (2.14)$$

where $a = |n - m| + 1 - 2/\beta$; $l_{\beta, m, a}$ is as in (2.11); and

$$d_{\beta, m, n} = \frac{\Gamma(\beta(n-m)/2) \Gamma(\beta/2)^m}{\Gamma(\beta n/2)}. \quad (2.15)$$

Proof. Let us first deal with $\beta = 1$ case, which by the discussion before Lemma 1 reduces to computing the spectral measures of a matrix H from $LOE_{(m, n)}$. For this ensemble, the eigenvalue distribution is as stated, since the nonzero eigenvalues of $LOE_{(m, n)}$ are distributed identically to the eigenvalues of $LOE_{(n, m)}$.

With probability one, we may assume that eigenvalues of H satisfy $\lambda_1 > \dots > \lambda_m > 0 = 0 = \dots = 0$ ($n - m$ zeros). Let us choose an orthonormal system of (real) eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ of H corresponding to these eigenvalues, respectively. We pick each \mathbf{u}_j at random uniformly from the set of all possible choices. Since for any $n \times n$ orthogonal matrix O , the matrix $O^T H O$ also belongs to $LOE_{(m, n)}$, we can see that: \mathbf{u}_1 is uniformly distributed on the unit sphere $\{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\| = 1\}$; and for any $1 \leq j \leq n$, the vector \mathbf{u}_j conditionally on $\mathbf{u}_1, \dots, \mathbf{u}_{j-1}$ is uniformly distributed on the subset of this unit sphere that is orthogonal to $\mathbf{u}_1, \dots, \mathbf{u}_{j-1}$. Therefore the matrix consisting of the eigenvectors as its columns is a Haar distributed orthogonal matrix (see, e.g., [KK, Prop 2.2(a)]). Then its first row (x_1, \dots, x_n) is distributed uniformly on the unit sphere $\{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\| = 1\}$. Now recalling (2.5), we obtain that $w_j = x_j^2$, $1 \leq j \leq m$, and $w_0 = x_{m+1}^2 + \dots + x_n^2$. Now one can apply arguments from the proof of [KN04, Cor A.2] (note that $dw_j = 2w_j^{1/2} dx_j$) to see that the joint distribution of w_1, \dots, w_m is proportional to $w_0^{(n-m-2)/2} \prod_{j=1}^m w_j^{-1/2} dw_1 \dots dw_m$. Let us ignore the normalization constant for now and come back to it in the end.

Just as in Dumitriu–Edelman [DE02], this allows us to compute the Jacobian of the change of variables from $\{x_j, y_j\}_{j=1}^m$ in (2.3) to $\{\lambda_j, w_j\}_{j=1}^m$. Before proceeding, we need to clarify why this change of variables is bijective. By Favard's theorem (see, e.g., [Sim11, Thms 1.3.2–1.3.3]), there is one-to-one correspondence between all $(m+1) \times (m+1)$ Jacobi matrices (2.1) with all $a_j > 0$ ($1 \leq j \leq m$) and all probability measures supported on $m+1$ distinct points. This trivially implies that there is one-to-one correspondence between all positive semi-definite $(m+1) \times (m+1)$ Jacobi matrices \mathcal{J} with $\det \mathcal{J} = 0$ and $a_j > 0$ (for each $1 \leq j \leq m$) and all probability measures supported on $m+1$ points of the form (2.12)–(2.13). By semi-definiteness, any such \mathcal{J} can be Cholesky factorized $\mathcal{J} = B^* B$ with B upper-triangular with non-negative entries on the diagonal. Since \mathcal{J} is tridiagonal, it is not hard to see that this $(m+1) \times (m+1)$ matrix B must be two-diagonal as in (2.3) with $x_j \geq 0$, $1 \leq j \leq m+1$. Since $\det \mathcal{J} = 0$, we must have that $x_j = 0$ for at least one

$1 \leq j \leq m+1$. But since all $a_j > 0$, we must necessarily have $x_{m+1} = 0$, and $x_j \neq 0$ for $1 \leq j \leq m$. $a_j > 0$ also implies that $y_j > 0$, $1 \leq j \leq m$. Conversely, any $(m+1) \times (m+1)$ matrix B of the form (2.3) with $x_j > 0, y_j > 0$ for $1 \leq j \leq m$ and $x_{m+1} = 0$ clearly leads to a positive semi-definite $(m+1) \times (m+1)$ Jacobi matrix \mathcal{J} with $\det \mathcal{J} = 0$ and all $a_j > 0$ ($1 \leq j \leq m$).

Using the matrix model in Lemma 2 (case $m < n$) and the distribution (2.14) that we proved for $\beta = 1$, we obtain that the Jacobian is proportional (let us ignore the normalizing constants for now) to

$$\begin{aligned} \det \frac{\partial(x_1, \dots, x_m, y_1, \dots, y_m)}{\partial(\lambda_1, \dots, \lambda_m, w_1, \dots, w_m)} &\propto \prod_{j=1}^m x_j^{-m+j} e^{x_j^2/2} \prod_{j=1}^m y_j^{-n+j+1} e^{y_j^2/2} \\ &\times w_0^{\frac{n-m}{2}-1} \prod_{j=1}^m w_j^{-\frac{1}{2}} \prod_{j=1}^m \lambda_j^{\frac{n-m-1}{2}} e^{-\frac{\lambda_j}{2}} \prod_{1 \leq j < k \leq m} |\lambda_j - \lambda_k| \end{aligned}$$

Now, recall Lemma 2. The joint distribution of $\{x_1, \dots, x_m, y_1, \dots, y_m\}$ for $L\beta E_{(m,n)}$, $m \leq n-1$, is, up to a normalizing constant,

$$\propto \prod_{j=1}^m x_j^{\beta(m-j-1)-1} e^{-x_j^2/2} dx_j \prod_{j=1}^m y_j^{\beta(n-j)-1} e^{-y_j^2/2} dy_j.$$

Using the above Jacobian, we obtain that this distribution becomes

$$\begin{aligned} &\propto \prod_{j=1}^m x_j^{(\beta-1)(m-j-1)} \prod_{j=1}^m y_j^{(\beta-1)(n-j)} \\ &\times w_0^{\frac{n-m}{2}-1} \prod_{j=1}^m w_j^{-\frac{1}{2}} \prod_{j=1}^m \lambda_j^{\frac{n-m-1}{2}} e^{-\frac{\lambda_j}{2}} \prod_{1 \leq j < k \leq m} |\lambda_j - \lambda_k| d\lambda_1 \dots d\lambda_m dw_1 \dots dw_m. \end{aligned} \quad (2.16)$$

Lemma 5. (i) *The following identity holds*

$$\prod_{j=1}^m x_j^{m-j+1} y_j^{m-j+1} = \prod_{j=0}^m w_j^{1/2} \prod_{1 \leq j < k \leq m} |\lambda_j - \lambda_k| \prod_{j=1}^m \lambda_j.$$

(ii) *The following identity holds*

$$\prod_{j=1}^m y_j^2 = w_0 \prod_{j=1}^m \lambda_j.$$

Proof. (i) follows immediately by noting that $x_j y_j = a_j$, $1 \leq j \leq m$, and then applying [DE02, Lemma 2.7]. Note the clash of notations: their n is our $m+1$, their $\{b_1, \dots, b_{n-1}\}$, $\{\lambda_1, \dots, \lambda_n\}$, and $\{q_1^2, \dots, q_n^2\}$ are ours $\{a_m, \dots, a_1\}$, $\{\lambda_1, \dots, \lambda_m, 0\}$, and $\{w_1, \dots, w_m, w_0\}$, respectively.

To prove (ii), we use theory of orthogonal polynomials, see, e.g., [Sim11]. By combining [Sim11, Prop 3.2.8] and [Sim11, Prop 2.3.12] we get

$$w_0 = -\lim_{z \rightarrow 0} \langle \mathbf{e}_1, z(\mathcal{J} - z)^{-1} \mathbf{e}_1 \rangle = \lim_{z \rightarrow 0} \frac{z q_{m+1}(z)}{p_{m+1}(z)} = \frac{q_{m+1}(0)}{p_{m+1}(0)},$$

where p_j 's and q_j 's are the orthonormal polynomials associated to \mathcal{J} of the first and second kind, respectively (in order to define p_{m+1} and q_{m+1} we need a_{m+1} which we take to be an arbitrary positive number). By [Sim11, Thm 1.2.4], $p_{m+1}(z) =$

$(\prod_{j=1}^{m+1} a_j^{-1}) \det(z - \mathcal{J})$, so $p'_{m+1}(0) = (-1)^m \prod_{j=1}^{m+1} a_j^{-1} \prod_{j=1}^m \lambda_j$. Using the Wronskian relation [Sim11, Prop 3.2.3] and $p_{m+1}(0) = 0$ (since 0 is an eigenvalue of \mathcal{J}), we obtain $q_{m+1}(0) = 1/(a_{m+1}p_m(0))$. Finally, $p_m(z) = (\prod_{j=1}^m a_j^{-1}) \det(z - \mathcal{J}_{m \times m})$, where $\mathcal{J}_{m \times m}$ is the $m \times m$ top left corner of \mathcal{J} . Recall that $\mathcal{J} = B^*B$. It is easy to see that $\mathcal{J}_{m \times m} = B_{m \times m}^* B_{m \times m}$, where $B_{m \times m}$ is the $m \times m$ top left corner of B . Therefore $p_m(0) = (\prod_{j=1}^m a_j^{-1}) \det(-B_{m \times m}^* B_{m \times m}) = (-1)^m (\prod_{j=1}^m a_j^{-1}) \prod_{j=1}^m x_j^2$. Combining this all together with $a_j = x_j y_j$, $1 \leq j \leq m$, we obtain (ii). \square

With the aid of this lemma we can now simplify the distribution (2.16). Indeed, using the identity (i) we can eliminate the product of involving x_j 's, and then using (ii), we can eliminate the product involving y_j 's. It is an easy exercise to see that, up to a normalization constant, this reduces (2.16) to (2.14). Finally, note that $l_{\beta, m, a}$ is the right normalization constant for the eigenvalues in (2.14) by Lemma 4. And the normalization constant $d_{\beta, m, n}$ can be computed by evaluating the Dirichlet integral, see, e.g., [KN04, Cor A.4]. \square

3. RANK ONE PERTURBATIONS

Let H be an $n \times n$ matrix from one of the six ensembles GOE_n , $LOE_{m \times n}$ (let us refer to these two ensembles as the $\beta = 1$ case throughout this section); GUE_n , $LUE_{m \times n}$ ($\beta = 2$ case); GSE_n , $LSE_{m \times n}$ ($\beta = 4$ case). Let

$$H_{eff} = H + i\Gamma, \quad (3.1)$$

where $\Gamma = (\Gamma_{jk})_{j,k=1}^n$ is an $n \times n$ positive matrix that is independent of H with real (if $\beta = 1$), complex (if $\beta = 2$), or quaternion (if $\beta = 4$) entries. We assume that Γ has rank 1 (for the case $\beta = 4$, the (right) rank is viewed over quaternions, see, e.g., [Rod14]).

Since Γ is Hermitian, we can diagonalize $\Gamma = U^*(lI_{1 \times 1})U$, where $I_{1 \times 1}$ is the $n \times n$ matrix with $(1,1)$ -entry equal to 1 and 0 everywhere else, and U is orthogonal, unitary, or unitary symplectic for $\beta = 1, 2, 4$, respectively (for quaternion diagonalization, see, e.g., [Rod14, Thm 5.3.6]). Since the Hilbert–Schmidt norm should be preserved, we see that $l = \|\Gamma\|_{HS} = (\sum_{j,k=1}^n |\Gamma_{jk}|^2)^{1/2}$.

Then $H_{eff} = U^*(UHU^* + ilI_{1 \times 1})U$, where U is independent of H . From Definitions 2 and 3, it is clear that UHU^* belongs to the same ensemble as H . Therefore we can apply the tridiagonalization procedure from Subsection 2.2 to reduce UHU^* to the Dumitriu–Edelman form: $UHU^* = S^* \mathcal{J} S$ with \mathcal{J} as in Lemmas 1 or 2, and S unitary with $S\mathbf{e}_1 = S^*\mathbf{e}_1 = \mathbf{e}_1$. This implies $SI_{1 \times 1}S^* = I_{1 \times 1}$ and therefore

$$H_{eff} = U^*S^*(\mathcal{J} + ilI_{1 \times 1})SU.$$

This shows that H_{eff} can be unitarily reduced to a tridiagonal form whose all entries are real, except for the complex $(1,1)$ entry. We can formalize it into a theorem.

Theorem 1 (Matrix model for rank one non-Hermitian perturbations of Gaussian and Wishart ensembles). *Let H be taken from one of the six ensembles GOE_n , GUE_n , GSE_n , $LOE_{m \times n}$, $LUE_{m \times n}$, $LSE_{m \times n}$. Suppose $\Gamma \geq \mathbf{0}$, $\text{rank } \Gamma = 1$, and Γ is independent of H . Then $H_{eff} = H + i\Gamma$ is unitarily equivalent to*

$$\mathcal{J} + ilI_{1 \times 1} \quad (3.2)$$

where \mathcal{J} is as in Lemma 1 or 2, respectively, and $l = \|\Gamma\|_{HS} = (\sum_{j,k=1}^n |\Gamma_{jk}|^2)^{1/2}$ is independent of \mathcal{J} .

Remark. 1. Just like Dumitriu–Edelman models, this tridiagonal matrix ensemble (3.2) makes sense for any $0 < \beta < \infty$, not merely $\beta = 1, 2, 4$. For the obvious reasons we will refer to these as non-Hermitian rank one perturbations of $G\beta E_n$ and $L\beta E_{(m,n)}$ ensembles.

4. JOINT EIGENVALUE DISTRIBUTION

For the rest of the paper let

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}. \quad (4.1)$$

4.1. Perturbations of Gaussian β -ensembles.

Theorem 2. *For any $0 < \beta < \infty$, let \mathcal{J} be from $G\beta E_n$ ensemble (see Lemma 1) and l be independent of \mathcal{J} distributed according to an absolutely-continuous probability distribution $F(l)dl$ on $(0, +\infty)$. Then the eigenvalues of (3.2) are distributed on $(\mathbb{C}_+)^n$ according to*

$$\begin{aligned} & \frac{1}{h_{\beta,n}} e^{-\frac{1}{2} \sum_{j=1}^n (\operatorname{Re} z_j)^2 - \sum_{j < k} (\operatorname{Im} z_j)(\operatorname{Im} z_k)} \\ & \times \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{j < k} |z_j - z_k|^2 \frac{F(l)}{l^{\frac{\beta n}{2}-1}} d^2 z_1 \dots d^2 z_n, \end{aligned} \quad (4.2)$$

where $l = \sum_{j=1}^n \operatorname{Im} z_j$, $d^2 z$ stands for the 2-dimensional complex Lebesgue measure, and

$$h_{\beta,n} = 2^{n(\beta/2-1)} g_{\beta,n} c_{\beta,n},$$

where $g_{\beta,n}$ and $c_{\beta,n}$ are as in (2.8).

Remark. In view of Theorem 1, distribution (4.2) with $\beta = 1, 2, 4$ is the eigenvalue distribution of rank one perturbations of GOE_n , GUE_n , GSE_n , respectively.

Proof. First of all, because the imaginary part of \mathcal{J} is positive, we know that each of the eigenvalues z_1, \dots, z_n lies in \mathbb{C}_+ . The result of Arlinskiĭ–Tsekanovskiĭ [AT06, Thm 5.1] says that the mapping

$$\{a_j\}_{j=1}^{n-1}, \{b_j\}_{j=1}^n, l \mapsto z_1, \dots, z_n \quad (4.3)$$

$$(0, \infty)^{n-1} \times \mathbb{R}^n \times (0, \infty) \rightarrow (\mathbb{C}_+)^n \quad (4.4)$$

is one-to-one and onto (up to permutations of z_j 's). Then so is the mapping $\{\lambda_j\}_{j=1}^n, \{w_j\}_{j=1}^{n-1}, l \mapsto z_1, \dots, z_n$, where μ (2.5) is the spectral measure of \mathcal{J} . Let us compute the Jacobian of this transformation.

Lemma 6.

$$\left| \det \frac{\partial (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_1, \dots, \operatorname{Im} z_n)}{\partial (\lambda_1, \dots, \lambda_n, w_1, \dots, w_{n-1}, l)} \right| = l^{n-1} \prod_{j < k} \frac{|\lambda_j - \lambda_k|^2}{|z_j - z_k|^2}. \quad (4.5)$$

Proof. Denote $\mathcal{J}_l = \mathcal{J} + ilI_{1 \times 1}$. Define $m(z) = \langle e_1, (\mathcal{J} - z)^{-1} e_1 \rangle = \sum_{j=1}^n \frac{w_j}{\lambda_j - z}$. Let $\sum_{j=0}^n \kappa_j z^j = \det(z - \mathcal{J}_l) = \prod_{j=1}^n (z - z_j)$, where $\kappa_n = 1$. Then

$$\begin{aligned} \prod_{j=1}^n (z - z_j) &= \sum_{j=0}^n \kappa_j z^j = \det(z - \mathcal{J} - ilI_{1 \times 1}) \\ &= \det(z - \mathcal{J}) \det(I - (z - \mathcal{J})^{-1} ilI_{1 \times 1}) = (1 + ilm(z)) \prod_{j=1}^n (z - \lambda_j). \end{aligned} \quad (4.6)$$

By taking the real parts we obtain

$$\frac{1}{2} \prod_{j=1}^n (z - z_j) + \frac{1}{2} \prod_{j=1}^n (z - \bar{z}_j) = \sum_{j=0}^n (\operatorname{Re} \kappa_j) z^j = \prod_{j=1}^n (z - \lambda_j), \quad (4.7)$$

which implies

$$\left| \det \frac{\partial (\operatorname{Re} \kappa_0, \dots, \operatorname{Re} \kappa_{n-1})}{\partial (\lambda_1, \dots, \lambda_n)} \right| = \prod_{j < k} |\lambda_j - \lambda_k|, \quad (4.8)$$

and

$$\frac{\partial (\operatorname{Re} \kappa_0, \dots, \operatorname{Re} \kappa_{n-1})}{\partial (w_1, \dots, w_{n-1}, l)} = \mathbf{0}, \quad (4.9)$$

the $n \times n$ zero matrix. Thus we just need to evaluate $\det \frac{\partial (\operatorname{Im} \kappa_0, \dots, \operatorname{Im} \kappa_{n-1})}{\partial (w_1, \dots, w_{n-1}, l)}$, regarding λ_j 's as constants.

The imaginary parts of (4.6) give

$$\begin{aligned} \sum_{j=0}^{n-1} (\operatorname{Im} \kappa_j) z^j &= l m(z) \prod_{j=1}^n (z - \lambda_j) = -l \sum_{j=1}^n w_j \prod_{\substack{1 \leq k \leq n \\ k \neq j}} (z - \lambda_k) \\ &= -l \left[\sum_{j=1}^{n-1} w_j (\lambda_j - \lambda_n) \prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (z - \lambda_k) \right] - l \prod_{k=1}^{n-1} (z - \lambda_k) \end{aligned} \quad (4.10)$$

Denote the polynomial in the square brackets as $s(z) = \sum_{j=0}^{n-2} s_j z^j$. The above equality implies

$$\det \frac{\partial (\operatorname{Im} \kappa_0, \dots, \operatorname{Im} \kappa_{n-1})}{\partial (s_0, \dots, s_{n-2}, l)} = (-1) \cdot (-l)^{n-1}. \quad (4.11)$$

Now note that $s(z)$ can trivially be rewritten as

$$s(z) = \sum_{j=1}^{n-1} \tilde{w}_j \prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} \frac{z - \lambda_k}{\lambda_j - \lambda_k},$$

where

$$\tilde{w}_j = w_j (\lambda_j - \lambda_n) \prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (\lambda_j - \lambda_k). \quad (4.12)$$

One can now recognize that $s(z)$ is the interpolating polynomial $s(\lambda_k) = \tilde{w}_k$ for $k = 1, \dots, n-1$. This implies

$$\left| \det \frac{\partial (\tilde{w}_1, \dots, \tilde{w}_{n-1})}{\partial (s_0, \dots, s_{n-2})} \right| = \prod_{1 \leq j < k \leq n-1} |\lambda_j - \lambda_k|. \quad (4.13)$$

Finally, from (4.12),

$$\det \frac{\partial (\tilde{w}_1, \dots, \tilde{w}_{n-1})}{\partial (w_1, \dots, w_{n-1})} = \prod_{j=1}^{n-1} (\lambda_j - \lambda_n) \prod_{1 \leq j < k \leq n-1} |\lambda_j - \lambda_k|^2. \quad (4.14)$$

Combining (4.11), (4.13), (4.14), we get

$$\left| \det \frac{\partial (\operatorname{Im} \kappa_0, \dots, \operatorname{Im} \kappa_{n-1})}{\partial (w_1, \dots, w_{n-1}, l)} \right| = l^{n-1} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|. \quad (4.15)$$

Using (4.8), (4.9), and the fact that the Jacobian of the transformation from $\operatorname{Re} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_1, \dots, \operatorname{Im} z_n$ to $\operatorname{Re} \kappa_0, \operatorname{Im} \kappa_0, \dots, \operatorname{Re} \kappa_{n-1}, \operatorname{Im} \kappa_{n-1}$ is equal to $\prod_{j < k} |z_j - z_k|^2$, we obtain

$$\left| \det \frac{\partial (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_1, \dots, \operatorname{Im} z_n)}{\partial (\lambda_1, \dots, \lambda_n, w_1, \dots, w_{n-1}, l)} \right| = l^{n-1} \prod_{j < k} \frac{|\lambda_j - \lambda_k|^2}{|z_j - z_k|^2}. \quad (4.16)$$

□

The joint distribution of $\{\lambda_j\}_{j=1}^n, \{w_j\}_{j=1}^{n-1}, l$ is

$$\frac{1}{g_{\beta, n} c_{\beta, n}} \prod_{j < k} |\lambda_j - \lambda_k|^\beta \prod_{j=1}^n e^{-\lambda_j^2/2} \prod_{j=1}^n w_j^{\beta/2-1} F(l) d\lambda_1 \dots d\lambda_n dw_1 \dots dw_{n-1} dl.$$

Using this and the Jacobian computation, we obtain that the distribution of z_j 's is

$$\frac{1}{g_{\beta, n} c_{\beta, n}} \prod_{j < k} |\lambda_j - \lambda_k|^{\beta-2} \prod_{j=1}^n e^{-\lambda_j^2/2} \prod_{j=1}^n w_j^{\beta/2-1} \frac{F(l)}{l^{n-1}} \prod_{j < k} |z_j - z_k|^2 d^2 z_1 \dots d^2 z_n. \quad (4.17)$$

Note that

$$l = -\operatorname{Im} \kappa_{n-1} = \sum_{j=1}^n \operatorname{Im} z_j, \quad (4.18)$$

$$\sum_{j=1}^n \lambda_j = \sum_{j=1}^n \operatorname{Re} z_j, \quad (4.19)$$

$$\sum_{j \neq k} \lambda_j \lambda_k = \sum_{j \neq k} \operatorname{Re}(z_j z_k). \quad (4.20)$$

The first equation comes from (4.10), while the latter two follow from (4.7). Then

$$\sum_{j=1}^n \lambda_j^2 = \left(\sum_{j=1}^n \operatorname{Re} z_j \right)^2 - \sum_{j \neq k} \operatorname{Re}(z_j z_k) = \sum_{j=1}^n (\operatorname{Re} z_j)^2 + 2 \sum_{j < k} (\operatorname{Im} z_j)(\operatorname{Im} z_k). \quad (4.21)$$

Finally, from (4.6),

$$-ilw_j = il \operatorname{Res}_{z=\lambda_j} m(z) = \operatorname{Res}_{z=\lambda_j} \prod_{k=1}^n \frac{z - z_k}{z - \lambda_k} = \frac{\prod_{k=1}^n (\lambda_j - z_k)}{\prod_{k \neq j} (\lambda_j - \lambda_k)}, \quad (4.22)$$

so

$$\prod_{j=1}^n w_j = \left(\frac{i}{l} \right)^n \frac{\prod_{j,k} (\lambda_j - z_k)}{\prod_{j < k} |\lambda_j - \lambda_k|^2} = \left(\frac{i}{l} \right)^n \frac{1}{2^n} \frac{\prod_{j,k} (\bar{z}_j - z_k)}{\prod_{j < k} |\lambda_j - \lambda_k|^2} = \frac{1}{(2l)^n} \frac{\prod_{j,k} |\bar{z}_j - z_k|}{\prod_{j < k} |\lambda_j - \lambda_k|^2}, \quad (4.23)$$

where we used (4.7) with $z = z_k$, $k = 1, \dots, n$. Combining (4.18), (4.21), (4.23) with (4.17), we obtain (4.2). □

Examples. (1) Since Γ in Theorem 1 has rank 1, we can decompose it as $\Gamma = L^*L$, where $L = (l_{1j})_{j=1}^n$ is an $1 \times n$ matrix. Assuming the entries l_{1j} of L are independent and normal $N(0, \sigma^2 \mathbf{I}_\beta)$, then $l = \sum_{j=1}^n |l_{1j}|^2 \sim \sigma^2 \chi_{\beta n}^2$, that is $F(l) = \frac{1}{(\sqrt{2}\sigma)^{\beta n} \Gamma(\beta n/2)} l^{\beta n/2-1} e^{-l/(2\sigma^2)}$. In this special case, distribution (4.2) becomes

$$\frac{1}{(\sqrt{2}\sigma)^{\beta n} \Gamma(\beta n/2) c_{\beta, n} g_{\beta, n}} \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\beta/2-1} \prod_{j < k} |z_j - z_k|^2 \times e^{-\frac{1}{2} \sum_{j=1}^n (\operatorname{Re} z_j)^2 - \sum_{j < k} (\operatorname{Im} z_j)(\operatorname{Im} z_k) - \frac{1}{2\sigma^2} \sum_{j=1}^n \operatorname{Im} z_j} d^2 z_1 \dots d^2 z_n. \quad (4.24)$$

agreeing with the formula obtained by Stöckman–Šeba [SŠ98, Eq (4.4)].

(2) If one instead takes, perhaps less naturally, $l \sim \chi_{\beta n/2}$, then the eigenvalue density simplifies to

$$\propto \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\beta/2-1} \prod_{j < k} |z_j - z_k|^2 e^{-\frac{1}{2} \sum_{j=1}^n |z_j|^2 - 2 \sum_{j < k} (\operatorname{Im} z_j)(\operatorname{Im} z_k)} d^2 z_1 \dots d^2 z_n.$$

4.2. Perturbations of Laguerre β -ensembles. Let us first address the question of which eigenvalue configurations are possible for rank one perturbations of Wishart or β -Laguerre ensembles. Unlike the Gaussian case which was easy due to the application of Arlinskiĭ–Tsekanovskii’s [AT06, Thm 5.1], here we perturb a positive (semi-) definite matrix.

Proposition 2. (i) Let

$$\mathcal{I}_l = \mathcal{J} + ilI_{1 \times 1}, \quad (4.25)$$

where $l > 0$ and \mathcal{J} is an $n \times n$ positive definite (real) Jacobi matrix (2.1) with $a_j > 0$, $j = 1, \dots, n-1$. Its eigenvalues, counting algebraic multiplicities, belong to

$$\left\{ (z_j)_{j=1}^n \in (\mathbb{C}_+)^n : \sum_{j=1}^n \operatorname{Arg} z_j < \frac{\pi}{2} \right\}. \quad (4.26)$$

Moreover, for every configuration of n points from (4.26) there exists a unique matrix \mathcal{I}_l of the form above with such a system of eigenvalues.

(ii) Let

$$\mathcal{I}_l = \mathcal{J} + ilI_{1 \times 1},$$

where $l > 0$ and \mathcal{J} is an $(m+1) \times (m+1)$ positive semi-definite (real) Jacobi matrix (2.1) with $a_j > 0$, $j = 1, \dots, m$, satisfying $\det \mathcal{J} = 0$. Its eigenvalues, counting with their algebraic multiplicities, belong to

$$\left\{ (z_j)_{j=1}^{m+1} \in (\mathbb{C}_+)^{m+1} : \sum_{j=1}^{m+1} \operatorname{Arg} z_j = \frac{\pi}{2} \right\}. \quad (4.27)$$

Moreover, for every configuration of $m+1$ points from (4.27) there exists a unique matrix \mathcal{I}_l of the form above with such a system of eigenvalues.

Proof. As before, let z_j ’s be the eigenvalues of \mathcal{I}_l ; let λ_j ’s and w_j ’s be the eigenvalues and eigenweights of the spectral measure of \mathcal{J} (which is of the form (2.5) with (2.9) for the case (i) and (2.12) with (2.13) for the case (ii)). By [AT06], $z_j \in \mathbb{C}_+$ for every j .

Consider now case (i). Equations (4.7) and (4.10) imply

$$\operatorname{Re} s_k(z_1, \dots, z_n) = s_k(\lambda_1, \dots, \lambda_n), \quad k = 1, 2, \dots, n; \quad (4.28)$$

$$\operatorname{Im} s_k(z_1, \dots, z_n) = l \sum_{j=1}^n w_j s_{k-1}(\{\lambda_t\}_{t \neq j}), \quad k = 1, 2, \dots, n, \quad (4.29)$$

respectively, where $s_0 := 1$, and s_k ($k \geq 1$) is the k -th elementary symmetric polynomial

$$s_k(z_1, \dots, z_n) := \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} z_{j_1} \dots z_{j_k}. \quad (4.30)$$

Since for each j , $\lambda_j > 0, w_j > 0, l > 0$, we obtain that z_1, \dots, z_n must belong to

$$\{(z_j)_{j=1}^n \in (\mathbb{C}_+)^n : s_k(z_1, \dots, z_n) \in Q_1, \quad k = 1, 2, \dots, n\}, \quad (4.31)$$

where $Q_1 := \{z : 0 < \operatorname{Arg} z < \pi/2\}$. Conversely, take a sequence of points from (4.31). Since this sequence belongs to $(\mathbb{C}_+)^n$, we know from [AT06, Thm 5.1] that there exists a unique matrix of the form $\mathcal{J} + ilI_{1 \times 1}$ with $l > 0$ and $a_j > 0$, $j = 1, \dots, n-1$. We claim that in fact \mathcal{J} is positive definite, that is, $\lambda_j > 0$ for all j . Indeed, equation (4.7) along with the positivity of (4.28) implies that $\lambda_1, \dots, \lambda_n$ are the real roots of the polynomial $\prod_{j=1}^n (z - \lambda_j)$ with alternating signs of the coefficients. By Descartes' rule of signs, we know such a polynomial cannot have negative zeros. This means that all λ_j 's are indeed positive. Therefore (4.31) is precisely the space of all possible eigenvalue configurations of H_{eff} . Let us now show that it coincides with (4.26).

It is elementary that (4.26) is a subset of (4.31). To see the converse, take any sequence from (4.31). Since $s_n(z_1, \dots, z_n) = z_1 z_2 \dots z_n \in Q_1$, we must have that

$$0 + 2k\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + \dots + \operatorname{Arg} z_n < \pi/2 + 2k\pi \quad (4.32)$$

for some integer $k \geq 0$. We already know that these z_1, \dots, z_n are the eigenvalues of $\mathcal{J} + ilI_1$, where \mathcal{J} is *positive definite*. Let us now fix \mathcal{J} and view z_1, \dots, z_n as functions of $l \geq 0$ only. Each of these functions is continuous and never passes through 0. For any $0 < l < \infty$, we have (4.32) for some k . But when $l = 0$ the sum of the arguments is zero. By continuity $k = 0$ for any l . This shows that (4.31) is a subset of (4.26), and therefore they coincide.

To deal with the case (ii), we use similar arguments with $m+1$ instead of n and $\lambda_1, \dots, \lambda_m, 0$ as the eigenvalues (with $\lambda_j > 0, j = 1, \dots, m$). With this in mind, equations (4.28) and (4.29) imply that the eigenvalues z_1, \dots, z_{m+1} of $\mathcal{J} + ilI_{1 \times 1}$ belong to

$$\begin{aligned} \{(z_j)_{j=1}^{m+1} \in (\mathbb{C}_+)^{m+1} : s_{m+1}(z_1, \dots, z_{m+1}) \in i\mathbb{R}_+; \\ s_k(z_1, \dots, z_{m+1}) \in Q_1, \quad k = 1, 2, \dots, m\}, \end{aligned} \quad (4.33)$$

where $\mathbb{R}_+ = \{z \in \mathbb{R} : z > 0\}$. Conversely, by [AT06, Thm 5.1], any configuration of point from (4.33) coincides with eigenvalues of some $\mathcal{J} + ilI_{1 \times 1}$, $l > 0$. One obtains that the eigenvalues $\lambda_1, \dots, \lambda_{m+1}$ of \mathcal{J} satisfies $s_k(\lambda_1, \dots, \lambda_{m+1}) > 0$ for $k = 1, \dots, m$ and $s_{m+1}(\lambda_1, \dots, \lambda_{m+1}) = 0$. This implies $\lambda_j > 0$ for all j except for one zero eigenvalue.

Finally, let us show that (4.33) coincides with (4.27). The inclusion (4.27) \subseteq (4.33) is easy. Conversely, take any configuration $\{z_j\}_{j=1}^{m+1}$ from (4.33). By the above, these points are the eigenvalues of some $\mathcal{J} + ilI_{1 \times 1}$ with $l > 0$, where \mathcal{J} has

eigenvalues $\{0, \lambda_1, \dots, \lambda_m\}$ with $\lambda_j > 0$ for $1 \leq j \leq m$. Since $s_{m+1} \in i\mathbb{R}_+$ in (4.33), we have

$$\text{Arg } z_1 + \text{Arg } z_2 + \dots + \text{Arg } z_{m+1} = \pi/2 + 2k\pi \quad (4.34)$$

for some integer $k \geq 0$. After reordering, we can assume that $z_j \rightarrow \lambda_j$, $1 \leq j \leq m$, and $z_{m+1} \rightarrow 0$ when $l \rightarrow 0$ (while \mathcal{J} is fixed). Therefore $\text{Arg } z_j \rightarrow 0$ as $l \rightarrow 0$ for $1 \leq j \leq m$, while $0 \leq \text{Arg } z_{m+1} \leq \pi/2$ for any l . This proves that $k = 0$, and so (4.33) \subseteq (4.27), finishing the proof. \square

The following may be known, but if not, it may be of interest on its own. Denote $\bar{\mathbb{C}}_+ := \{z : \text{Im } z \geq 0\}$.

Corollary 1. *Let*

$$H_{eff} = H + i\Gamma,$$

where H and Γ are positive semi-definite with $\text{rank } \Gamma \leq 1$. The eigenvalues of H_{eff} , counting with their algebraic multiplicities, belong to

$$\left\{ (z_j)_{j=1}^n \in (\bar{\mathbb{C}}_+)^n : \sum_{j=1}^n \text{Arg } z_j \leq \frac{\pi}{2} \right\},$$

and every such a configuration may occur.

Remarks. 1. We stress that this is deterministic result.

2. We adopt the convention $\text{Arg } 0 = 0$ here.

3. Using our methods one can prove a similar statement for the case when H is not positive-semidefinite, but has s negative eigenvalues. The eigenvalues of H_{eff} then belong to $\{(z_j)_{j=1}^n \in (\bar{\mathbb{C}}_+)^n : \frac{\pi}{2} + \pi(s-1) < \sum_{j=1}^n \text{Arg } z_j \leq \frac{\pi}{2} + \pi s\}$.

Proof. Just as in Section 3, we can tridiagonalize $H + i\Gamma = V^*(\mathcal{J} + ilI_{1 \times 1})V$, where V is unitary, $l > 0$, and \mathcal{J} some positive semi-definite tridiagonal $n \times n$ matrix (2.1). Then just apply the previous proposition. Note that some of the a_j 's might be zero which is why we obtain non-strict inequalities in $0 \leq \text{Arg } z_j \leq \frac{\pi}{2}$. \square

Now that we know the possible configurations of the eigenvalues, we can compute their joint distribution.

Theorem 3. *For any $0 < \beta < \infty$ and any integer $m, n > 0$, let \mathcal{J} be the $n \times n$ matrix from $L\beta E_{(m,n)}$ ensemble (see Subsection 2.4) and l be independent of \mathcal{J} distributed according to an absolutely-continuous probability distribution $F(l)dl$ on $(0, +\infty)$.*

(i) *If $m \geq n$, then the eigenvalues $\{z_1, \dots, z_n\}$ of $\mathcal{J}_l = \mathcal{J} + ilI_{1 \times 1}$ are distributed on (4.26) according to*

$$\frac{1}{q_{\beta,n,a}} \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{j < k} |z_j - z_k|^2 e^{-\frac{1}{2} \sum_{j=1}^n \text{Re } z_j} \left(\text{Re } \prod_{j=1}^n z_j \right)^{\frac{\beta a}{2}} \frac{F(l)}{l^{\frac{\beta n}{2}-1}} d^2 z_1 \dots d^2 z_n, \quad (4.35)$$

where $l = \sum_{j=1}^n \text{Im } z_j$, $a = |m - n| + 1 - 2/\beta$, and

$$q_{\beta,n,a} = 2^{n(\beta/2-1)} l_{\beta,n,a} c_{\beta,n},$$

where $l_{\beta,n,a}$ and $c_{\beta,n}$ are as in (2.11) and (2.8).

(ii) *If $m \leq n-1$, then eigenvalues of $\mathcal{J}_l = \mathcal{J} + ilI_{1 \times 1}$ are $\{z_1, \dots, z_{m+1}, 0, \dots, 0\}$ with $\{z_1, \dots, z_{m+1}\} =: \{r_1 e^{i\theta_1}, \dots, r_{m+1} e^{i\theta_{m+1}}\}$ distributed on (4.27) according to*

$$\frac{1}{t_{\beta,m,n}} \prod_{j,k=1}^{m+1} |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{1 \leq j < k \leq m+1} |z_j - z_k|^2 e^{-\frac{1}{2} \sum_{j=1}^{m+1} \operatorname{Re} z_j} \prod_{j=1}^{m+1} |z_j|^{\frac{\beta(n-m-1)}{2}} \frac{F(l)}{l^{\frac{\beta n}{2}-1}} dr_1 \dots dr_{m+1} d\theta_1 \dots d\theta_m, \quad (4.36)$$

where $l = \sum_{j=1}^{m+1} \operatorname{Im} z_j$ and

$$t_{\beta,m,n} = (m+1)2^{(m+1)(\beta/2-1)} l_{\beta,m,a} d_{\beta,m,n}, \quad (4.37)$$

where $a = |n-m| + 1 - 2/\beta$, and $l_{\beta,m,a}$ and $d_{\beta,m,n}$ are as in (2.11) and (2.15).

Remarks. 1. In view of Theorem 1, distributions (4.35) and (4.36) with $\beta = 1, 2, 4$ are the eigenvalue distribution of rank one perturbations of the Wishart ensembles $LOE_{(m,n)}$, $LUE_{(m,n)}$, $LSE_{(m,n)}$, respectively.

2. In (ii), $\theta_{m+1} = \pi/2 - \sum_{j=1}^m \theta_j$ is implicit due to (4.27).

Proof. (i) We can take the known joint distribution of the eigenvalues λ_j 's, eigenweights w_j 's (see Lemma 4), and l and change the variables to z_j 's (by Proposition 2(i) it is one-to-one and onto (4.26), so the Jacobian (4.16) applies). Using (4.23), (4.18), (4.19), (4.28) (with $k = n$), we obtain the resulting distribution (4.35).

(ii) By Proposition 2(ii), the map from the spectral measures of the form (2.12)–(2.13) to the eigenvalues of $\mathcal{J} + iI_{1 \times 1}$: $\lambda_1, \dots, \lambda_m, w_1, \dots, w_m, l \mapsto z_1, \dots, z_{m+1}$ is one-to-one and onto (4.27) (if we impose some natural ordering on λ_j 's and z_j 's; we will remove it in the end of the proof). Its Jacobian is of course different from (4.16) computed earlier. Similar to the notation in the proof of Lemma 6, let $m(z) = \langle e_1, (\mathcal{J} - z)^{-1} e_1 \rangle = -\frac{w_0}{z} + \sum_{j=1}^m \frac{w_j}{\lambda_j - z}$ and $\sum_{j=0}^{m+1} \kappa_j z^j = \det(z - \mathcal{J}_l) = \prod_{j=1}^{m+1} (z - z_j)$, where $\kappa_{m+1} = 1$. Because of $\det \mathcal{J} = 0$, we obtain $\operatorname{Re} \kappa_0 = 0$. Following similar reasoning as in the proof of Lemma 6, we first obtain the value of the Jacobian

$$\left| \det \frac{\partial (\operatorname{Re} \kappa_1, \dots, \operatorname{Re} \kappa_m, \operatorname{Im} \kappa_0, \dots, \operatorname{Im} \kappa_m)}{\partial (\lambda_1, \dots, \lambda_m, w_1, \dots, w_m, l)} \right| = l^m \prod_{j=1}^m \lambda_j \prod_{1 \leq j < k \leq m} |\lambda_j - \lambda_k|^2. \quad (4.38)$$

Now let $z_j = r_j e^{i\theta_j}$ be the polar decomposition of z_j . Since $\operatorname{Re}(z_1 \dots z_{m+1}) = (-1)^{m+1} \operatorname{Re} \kappa_0 = 0$, we have that $e^{i\theta_{m+1}}$ is determined by z_1, \dots, z_m . Therefore we have a one-to-one map $\mathbb{R}^{2m+1} \rightarrow \mathbb{R}^{2m+1}$ taking $r_1, \dots, r_{m+1}, \theta_1, \dots, \theta_m$ to $\operatorname{Re} \kappa_1, \dots, \operatorname{Re} \kappa_m, \operatorname{Im} \kappa_0, \dots, \operatorname{Im} \kappa_m$. We are headed towards computing its Jacobian on the manifold $\operatorname{Re}(z_1 \dots z_{m+1}) = 0$.

First off, it is trivial to see that on $\operatorname{Re}(z_1 \dots z_{m+1}) = 0$:

$$\left| \det \frac{\partial (\operatorname{Re} z_1, \dots, \operatorname{Re} z_m, \operatorname{Im} z_1, \dots, \operatorname{Im} z_m, \operatorname{Im} \kappa_0)}{\partial (r_1, \dots, r_m, \theta_1, \dots, \theta_m, r_{m+1})} \right| = \prod_{j=1}^m |z_j|^2. \quad (4.39)$$

We are left with computing the Jacobian of the $\mathbb{R}^{2m+1} \rightarrow \mathbb{R}^{2m+1}$ mapping $\operatorname{Re} z_1, \dots, \operatorname{Re} z_m, \operatorname{Im} z_1, \dots, \operatorname{Im} z_m, \operatorname{Im} \kappa_0 \mapsto \operatorname{Re} \kappa_1, \operatorname{Re} \kappa_m, \operatorname{Im} \kappa_1, \dots, \operatorname{Im} \kappa_m, \operatorname{Im} \kappa_0$ restricted to $\operatorname{Re} \kappa_0 = 0$, which is easily seen to be equal (cf., e.g., [KK, Lemma D.1]) to the Jacobian of the $\mathbb{C}^{2m} \times \mathbb{R} \rightarrow \mathbb{C}^{2m} \times \mathbb{R}$ map $z_1, \bar{z}_1, \dots, z_m, \bar{z}_m, \operatorname{Im} \kappa_0 \mapsto \kappa_1, \bar{\kappa}_1, \dots, \kappa_m, \bar{\kappa}_m, \operatorname{Im} \kappa_0$ restricted to $\operatorname{Re} \kappa_0 = 0$, where we treat z_j and \bar{z}_j as independent variables. In the notation (4.30), we have that $(-1)^{m+1-j} \kappa_j = s_{m+1-j}(z_1, \dots, z_{m+1})$

for $0 \leq j \leq m$. Using $\kappa_0 = (-1)^{m+1} z_1 \dots z_{m+1}$ and the trivial equality $s_{m-j}(y_1, \dots, y_m) = s_m(y_1, \dots, y_m) s_j(\frac{1}{y_1}, \dots, \frac{1}{y_m})$, we can write for $1 \leq j \leq m$,

$$(-1)^{m+1-j} \kappa_j = s_{m+1-j}(z_1, \dots, z_m) + (-1)^{m+1} \kappa_0 s_j(\frac{1}{z_1}, \dots, \frac{1}{z_m}). \quad (4.40)$$

Since $\bar{\kappa}_0 = -\kappa_0$, we also get

$$(-1)^{m+1-j} \bar{\kappa}_j = s_{m+1-j}(\bar{z}_1, \dots, \bar{z}_m) - (-1)^{m+1} \kappa_0 s_j(\frac{1}{\bar{z}_1}, \dots, \frac{1}{\bar{z}_m}). \quad (4.41)$$

These equalities imply that for $1 \leq j \leq m$ and $1 \leq s \leq m$,

$$\begin{aligned} \frac{\partial \kappa_j}{\partial z_s} &= (-1)^{m+1-j} s_{m-j}(z_1, \dots, \hat{z}_s, \dots, z_m) - (-1)^j \frac{\kappa_0}{z_s^2} s_{j-1}(\frac{1}{z_1}, \dots, \frac{1}{z_s}, \dots, \frac{1}{z_m}), \\ \frac{\partial \bar{\kappa}_j}{\partial \bar{z}_s} &= (-1)^{m+1-j} s_{m-j}(\bar{z}_1, \dots, \hat{\bar{z}}_s, \dots, \bar{z}_m) + (-1)^j \frac{\kappa_0}{\bar{z}_s^2} s_{j-1}(\frac{1}{\bar{z}_1}, \dots, \frac{1}{\bar{z}_s}, \dots, \frac{1}{\bar{z}_m}), \\ \frac{\partial \kappa_j}{\partial \bar{z}_s} &= \frac{\partial \bar{\kappa}_j}{\partial z_s} = 0, \end{aligned}$$

where \hat{v} means that a variable v is omitted. Using $s_{m-1}(y_1, \dots, y_{m-1}) s_{j-1}(\frac{1}{y_1}, \dots, \frac{1}{y_{m-1}}) = s_{m-j}(y_1, \dots, y_{m-1})$ again, we can rewrite

$$\begin{aligned} \frac{\partial \kappa_j}{\partial z_s} &= \frac{z_s - z_{m+1}}{z_s} (-1)^{m+1-j} s_{m-j}(z_1, \dots, \hat{z}_s, \dots, z_m), \\ \frac{\partial \bar{\kappa}_j}{\partial \bar{z}_s} &= \frac{\bar{z}_s - \bar{z}_{m+1}}{\bar{z}_s} (-1)^{m+1-j} s_{m-j}(\bar{z}_1, \dots, \hat{\bar{z}}_s, \dots, \bar{z}_m). \end{aligned}$$

Having this in hand, it is easy to write the Jacobian and perform a straightforward Gaussian elimination to arrive to

$$\left| \det \frac{\partial (\kappa_1, \bar{\kappa}_1, \dots, \kappa_m, \bar{\kappa}_m, \text{Im } \kappa_0)}{\partial (z_1, \bar{z}_1, \dots, z_m, \bar{z}_m, \text{Im } \kappa_0)} \right| = \prod_{j=1}^m |z_j|^{-2} \prod_{1 \leq j < k \leq m+1} |z_j - z_k|^2. \quad (4.42)$$

Combining this with (4.38) and (4.39), we get

$$\left| \det \frac{\partial (r_1, \dots, r_m, \theta_1, \dots, \theta_m, r_{m+1})}{\partial (\lambda_1, \dots, \lambda_m, w_1, \dots, w_m, l)} \right| = l^m \prod_{j=1}^m |\lambda_j| \frac{\prod_{1 \leq j < k \leq m} |\lambda_j - \lambda_k|^2}{\prod_{1 \leq j < k \leq m+1} |z_j - z_k|^2}. \quad (4.43)$$

Repeating the arguments from (4.22) and (4.23), we obtain

$$w_0 = \frac{\prod_{j=1}^{m+1} |z_j|}{l \prod_{j=1}^m |\lambda_j|}, \quad \text{and} \quad \prod_{j=1}^m w_j = \frac{1}{l^m 2^{m+1}} \frac{\prod_{j,k=1}^{m+1} |z_j - \bar{z}_k|}{\prod_{j=1}^{m+1} |z_j| \prod_{j=1}^m |\lambda_j| \prod_{j < k} |\lambda_j - \lambda_k|^2}.$$

Finally, just as in (i), we still have $\sum_{j=1}^m \lambda_j = \sum_{j=1}^{m+1} \text{Re } z_j$ and $l = \sum_{j=1}^{m+1} \text{Im } z_j$. Now, starting from the joint distribution of $\lambda_1, \dots, \lambda_m, w_1, \dots, w_m$ (see Proposition 1) and l , applying the Jacobian (4.43), and using these substitutions (note that terms with $\prod |\lambda_j|$ cancel out in the process), we arrive at the distribution (4.36). Note that the factor $(m+1)$ in (4.37) comes from removing the ordering of z_j 's and λ_j 's (there are $(m+1)!$ of permutations for $\{z_j\}_{j=1}^{m+1}$, and only $m!$ for $\{\lambda_j\}_{j=1}^m$). \square

Example. Choosing $F(l) \propto l^{\frac{\beta n}{2}-1} e^{-l/2}$ (as in Example (1) of the previous section, this is natural since corresponds to each entry of L (where $\Gamma = L^*L$) being normal), the distribution (4.35) becomes

$$\propto \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{j < k} |z_j - z_k|^2 e^{-\frac{1}{2} \sum_{j=1}^n (\operatorname{Re} z_j + \operatorname{Im} z_j)} \left(\operatorname{Re} \prod_{j=1}^n z_j \right)^{\frac{\beta a}{2}} d^2 z_1 \dots d^2 z_n,$$

and similar simplification can be made for the distribution (4.36).

REFERENCES

- [AT06] Yu. Arlinskii and E. Tsekanovskii, *Non-self-adjoint Jacobi matrices with a rank-one imaginary part*, J. Funct. Anal. **241** (2006), no. 2, 383–438. MR 2271925 (2007g:47041)
- [DE02] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), no. 11, 5830–5847. MR 1936554 (2004g:82044)
- [FK99] Y. V. Fyodorov and B. A. Khoruzhenko, *Systematic analytical approach to correlation functions of resonances in quantum chaotic scattering*, Phys. Rev. Lett. **83** (1999), no. 1, 65–68.
- [FS96] Y. V. Fyodorov and H.-J. Sommers, *Statistics of S-matrix poles in few-channel chaotic scattering: crossover from isolated to overlapping resonances*, JETP Letters **63** (1996), no. 12, 1026–1030.
- [FS97] ———, *Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering: random matrix approach for systems with broken time-reversal invariance*, J. Math. Phys. **38** (1997), no. 4, 1918–1981, Quantum problems in condensed matter physics. MR 1450906 (98k:81297)
- [FS03] ———, *Random matrices close to Hermitian or unitary: overview of methods and results*, J. Phys. A **36** (2003), no. 12, 3303–3347, Random matrix theory. MR 1986421
- [FS11] Y. V. Fyodorov and D. V. Savin, *Resonance scattering of waves in chaotic systems*, The Oxford handbook of random matrix theory, Oxford Univ. Press, Oxford, 2011, pp. 703–722. MR 2932654
- [KK] R. Killip and R. Kozhan, *Matrix models and eigenvalue statistics for truncations of classical ensembles of random unitary matrices*, (under submission, arXiv:1501.05160).
- [KN04] R. Killip and I. Nenciu, *Matrix models for circular ensembles*, Int. Math. Res. Not. (2004), no. 50, 2665–2701. MR 2127367 (2006h:82003)
- [MRW10] G. E. Mitchell, A. Richter, and H. A. Weidenmüller, *Random matrices and chaos in nuclear physics: nuclear reactions*, Rev. Modern Phys. **82** (2010), no. 4, 2845–2901. MR 2770945 (2012d:81351)
- [OW] S. O’Rourke and P. M. Wood, *Spectra of nearly hermitian random matrices*, (preprint arXiv:1510.00039).
- [Roc] J. Rochet, *Complex outliers of hermitian random matrices*, (preprint arXiv:1507.00455).
- [Rod14] L. Rodman, *Topics in quaternion linear algebra*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2014. MR 3241695
- [SFT99] H.-J. Sommers, Y. V. Fyodorov, and M. Titov, *S-matrix poles for chaotic quantum systems as eigenvalues of complex symmetric random matrices: from isolated to overlapping resonances*, J. Phys. A **32** (1999), no. 5, L77–L87. MR 1674416
- [Sim11] B. Simon, *Szegő’s theorem and its descendants: spectral theory for l^2 perturbations of orthogonal polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011. MR 2743058 (2012b:47080)
- [SŠ98] H.-J. Stöckmann and P. Šeba, *The joint energy distribution function for the Hamiltonian $H = H_0 - iWW^+$ for the one-channel case*, J. Phys. A **31** (1998), no. 15, 3439–3448. MR 1625050 (99b:81070)
- [SZ89] V. V. Sokolov and V. G. Zelevinsky, *Dynamics and statistics of unstable quantum states*, Nuclear Phys. A **504** (1989), no. 3, 562–588.
- [Ull69] N. Ullah, *On a generalized distribution of the poles of the unitary collision matrix*, J. Mathematical Phys. **10** (1969), 2099–2103.

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